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ASYMPTOTIC SUCTION PROFILES IN FREE CONVECTION LAMINAR BOUNDARY LAYER FLOWS

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INTRODUCTION

There are exact asymptotic solutions to several problems of laminar boundary layer flow over porous surfaces with uniform suction. The earliest, and one of the simplest recorded, is that of Griffith and Meredith [1] for two-dimensional incompressible flow over a semi-infinite flat plate. The asymptotic solution to the corresponding problem of compressible flow without heat transfer was obtained by Young [2], and this has been extended to flow with heat transfer by Lew and Fannucci [3].

Axially symmetric flows with uniform suction have also received some attention. Wuest [4], Lew [5] and Yasuhara [6] showed (independently) that for an incompressible flow along a circular cylinder, the axial component of velocity, w, has the simple asymptotic form

$$w = W[1 - (a/r)^R], \tag{1}$$

where a is the cylinder radius, r denotes distance from the axis, W is the axial component of velocity at large distances and

$$R = Va/\nu \tag{2}$$

is the Reynolds number associated with the suction velocity V, the radius of the cylinder and the kinematic viscosity ν. In a further contribution, Lew [7] extended the solution to compressible flow with heat transfer.

Flows involving free convection and suction appear to have received little attention. Eichhorn [8] obtained similarity solutions for free convective flow along a vertical flat plate with suitable non-uniform suction velocity and surface temperature. The only other contribution of this kind known to the present author is that of Kaloni [9] who obtained the asymptotic solution to a problem of flow of a visco-elastic fluid over a flat plate.

In each problem mentioned above, the asymptotic state is reached when vorticity generated at the boundary and diffusing away from it is exactly balanced by convection of vorticity towards the boundary brought about by the suction. For the steady asymptotic laminar state to be possible, the only condition on the suction velocity is that it should be positive.

This note is concerned largely with the problem of suction through the axially symmetric laminar boundary layer on a heated vertical circular cylinder, the axial component of flow being generated by natural convection. An interesting fact which emerges from the solution is that for a steady asymptotic laminar state to be possible it is not sufficient for the suction velocity, V, to be positive: it is also necessary to have $V > 2\kappa/a$, where κ is the thermal diffusivity and a is the cylinder radius.

ANALYSIS

Consider a semi-infinite circular cylinder, of radius a, placed with its axis vertical and its leading edge lowermost in a fluid of infinite extent. Let the cylinder be maintained at a uniform constant temperature T_1 and denote the ambient fluid temperature by T_0 ($< T_1$).

In addition to the natural convective flow which develops, suppose that there is a forced flow towards the cylinder due to extraction of fluid uniformly over the surface with velocity

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V. It will be assumed that at large distances from the leading edge of the cylinder the flow tends to a steady asymptotic laminar state. The range of values of the non-dimensional parameters in the problem consistent with this assumption will be determined *a posteriori*.

If r denotes distance from the cylinder axis, T denotes temperature in the fluid and v, w are, respectively, the outward radial component and upward axial component of velocity, the boundary layer equations governing the asymptotic flow are

$$v\frac{\mathrm{d}w}{\mathrm{d}r} = \beta g(T - T_0) + \frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right), \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(vr) = 0 \tag{4}$$

and

$$v\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{\kappa}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right); \tag{5}$$

the kinematic viscosity, thermal diffusivity and coefficient of expansion of the fluid have been assumed constant and are denoted by ν, κ, β respectively; g denotes the acceleration due to gravity. We have also assumed that viscous dissipation of energy is negligible and have taken into account variations in density only insofar as is necessary in the buoyancy term in equation (3). The boundary conditions are:

$$v = -V,\ w = 0,\ T = T_1 \quad \text{at} \quad r = a; \tag{6}$$

$$w \to 0,\ T \to T_0 \quad \text{as} \quad r \to \infty. \tag{7}$$

Equations (3)–(5) subject to the above boundary conditions are readily solved.

From (4) and (6) we have

$$v = -Va/r. \tag{8}$$

The solution of (5) for the temperature is then found to be

$$\frac{T - T_0}{T_1 - T_0} = \left(\frac{a}{r}\right)^{Pe}, \tag{9}$$

where

$$Pe = Va/\kappa > 0 \tag{10}$$

is the suction Péclét number.

Substituting for $T - T_0$ into (3) gives an elementary differential equation for w whose solution can only fit the boundary conditions (6) and (7) if

$$R > 0 \quad \text{and} \quad Pe > 2, \tag{11}$$

where R is the suction Reynolds number defined by (2). When these conditions hold, the solution may be expressed as

$$\frac{w}{V} = \frac{Gr}{R(Pe - 2)(R - Pe + 2)}\left[\left(\frac{a}{r}\right)^{Pe-2} - \left(\frac{a}{r}\right)^{R}\right], \tag{12}$$

where

$$Gr = \beta g(T_1 - T_0)a^3/\nu^2 \tag{13}$$

is the Grashof number and we have assumed that $R \neq Pe - 2$. In the particular case when $R = Pe - 2$, the solution takes the form

$$\frac{w}{V} = \frac{Gr}{R^2}\left(\frac{a}{r}\right)^{R}\log\frac{r}{a}. \tag{14}$$

The rate of heat transfer from the cylinder may be conveniently expressed in non-dimensional form in terms of the Nusselt number, defined by

$$Nu = q/[k(T_1 - T_0)], \tag{15}$$

where q is the rate of heat transfer per unit length of cylinder and k is the thermal conductivity of the fluid. Using (9) leads to the simple expression

$$Nu = 2\pi\, Pe. \tag{16}$$

CONCLUDING REMARKS

The shape of the velocity profile in a typical case ($R = Pe = 3$) is shown in Fig. 1. It may be observed that the profile rises steeply near the cylinder until the maximum value is reached, but the approach to the asymptotic value (zero) as r increases is relatively slow. The maximum point moves nearer to the cylinder as R increases.

For a steady asymptotic laminar state to be possible, we have seen that it is essential to have $R > 0$ and $Pe > 2$. The first condition implies that a radial flow of vorticity *towards* the boundary is necessary, although the suction for this purpose can be made arbitrarily small. The second condition indicates that it is also necessary for heat to be convected towards the boundary and for this purpose the suction must be maintained above a certain minimum (non-zero) level.

If there is a forced flow in the axial direction superimposed on that due to natural convection, the resultant velocity distribution may be obtained by adding expression (1) for w to that given by (12) or (14); this follows from the fact that the differential equation and boundary conditions for w are linear. The displacement area, defined by

$$\Delta = \int_a^\infty \left(1 - \frac{w}{W}\right)2\pi r\,\mathrm{d}r, \tag{17}$$

is of some interest in this case. Physically, this quantity may be regarded as the asymptotic value of the area through which the outermost stream surfaces are displaced due to thickening of the boundary layer with increasing downstream distance. In the absence of free convection, Δ is finite only for $R > 2$ (see [4–6]). With combined free convection, it is readily shown that, for finite Δ, the condition $Pe > 4$ must also be satisfied.

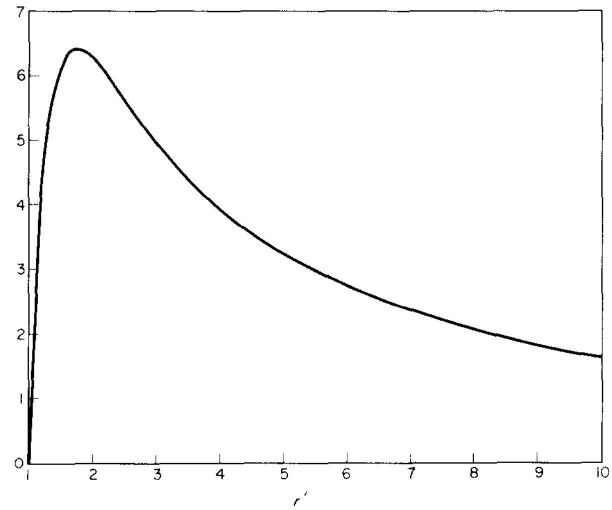

FIG. 1. Velocity profile in the case $R = Pe = 3$.
$(w' = 100w/Gr\,V;\ r' = r/a)$

Finally, we may observe that when $a \to \infty$, expressions (12) and (14) have the limiting forms

$$w = \frac{\beta g(T_1 - T_0)\,\kappa^2}{V^2(\nu - \kappa)}\left[e^{-Vy/\nu} - e^{-Vy/\kappa}\right] \qquad (\nu \neq \kappa) \qquad (18)$$

and

$$w = [\beta g(T_1 - T_0)/V]\, y\, e^{-Vy/\nu} \qquad (\nu = \kappa), \qquad (19)$$

where y denotes distance from the surface. These are the asymptotic profiles for a free convection boundary layer, with uniform suction, on a vertical flat plate; they can, of course, also be derived directly or as a special case of the work of Kaloni [9].

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